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A note on theoretical acoustical sources in motion

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The mathematical representation of acoustical sources in motion relative to the surrounding fluid is discussed. It is observed that several types of moving sources exist, and that it is sometimes necessary to choose the proper type. One of these sources currently appears to be more physically realistic than the others.

Introduction

The acoustical point source at rest in an infinite homogeneous fluid is very familiar. It requires no comment here except to state that it resembles an expanding and contracting spherical balloon.

The acoustical point source in motion relative to the immediately surrounding fluid is not so generally understood, since several types exist. This can readily be seen as follows. The time derivative of the velocity potential of a moving point source satisfies the wave equation. Relabelled as a potential this designates another type of moving point source. Thus infinitely many types exist. (The same operation performed on a stationary source is trivial, yielding the original potential.)

The distinction between different types of moving sources may be unimportant if the source is ultimately excluded from the flow and replaced by the velocities and pressures of its near field. In this way an inner boundary condition can be properly satisfied, and a consistent far field obtained. However, there are cases where it is useful and even essential to know the exact nature of the source.

In what follows we will discuss specifically simple harmonic sources in unaccelerated motion, though the ideas may carry over to more general cases.

Conventional moving sources

The conventional moving source (see, for example, Garrick 1957) can be represented as an array of stationary sources along the line of simulated motion. These sources are turned on and off in sequence, which gives the illusion of motion though no source actually moves. The fluid which is introduced at one point along the line of motion is left at rest with respect to the surrounding fluid at that point. It is the failure to transport this introduced fluid at the simulated velocity of the source that prevents the conventional moving source from corresponding to an expanding and contracting balloon in motion.

Assume that such a source is placed in an idealized circular cylindrical jet of uniform velocity and infinite length,† and held stationary relative to the air outside the jet (see figure 1). Then as the frequency approaches zero the far-field mean-square pressures approach the form shown in figure 1.

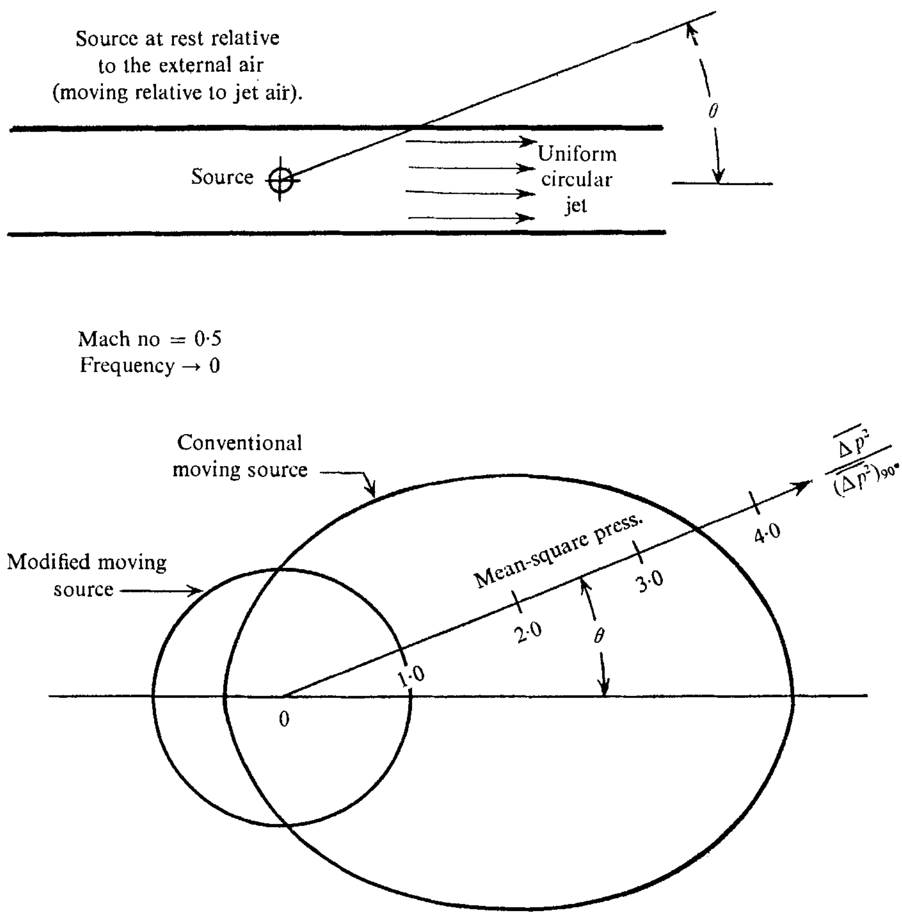

FIGURE 1. Far-field mean-square pressures for sources in a jet (low frequency limit).

Modified moving sources‡

In attempting to construct the conventional moving source theoretically, the authors observed (and presumably were not the first to do so) that other types of moving sources could be constructed. From periodic distributions of source strength on a plane, point sources were created by a Fourier integral process (see appendix). The periodic distributions were chosen to conform to standing waves in a co-ordinate system moving at the chosen source velocity. When these standing

† This type of problem was considered earlier by Gottlieb (1959) and by Moretti & Slutsky (1959).

‡ See also Graham (1968) and Graham & Graham (1969).

waves were waves of fluid particle *velocity* normal to the plane, the conventional moving source was obtained. However when the standing waves were waves of fluid particle *displacement* normal to the plane, which seems physically more significant, the modified source appeared.

The standing waves of fluid particle displacement suggest that the modified moving source carries its own fluid with it, unlike the conventional source. This idea is apparently verified by the following type of analysis. Consider a simple source drifting with the fluid in a uniform jet of infinite length. Since this source is stationary with respect to the immediately surrounding fluid its nature is quite unambiguous. Let the jet thickness approach zero (or the frequency approach zero), then it can be shown mathematically that the modified moving source results. Such a source is carrying its fluid with it, and so behaving as an expanding and contracting balloon† moving through the ambient air.

Assume that this modified moving source is placed in an idealized circular cylindrical jet of uniform velocity and infinite length, and held stationary relative to the air outside the jet (see figure 1). Then as the frequency approaches zero the far-field mean-square pressures (see figure 1) are independent of angle, and the behaviour is that of a simple stationary source in a homogeneous medium of infinite extent.

Comparison of theory with experiment

Since the far-field mean-square pressures are widely different for the conventional and the modified moving sources, an experimentally created source (see Atvars *et al.* 1965, for example) must be identified as to type. Is it one or the other of these, or a combination of them, or possibly a third type? Until this ambiguity is resolved, no useful comparison of theory with experiment can be made. In particular, statements in the literature that refraction theory for infinitely long jets exaggerates refraction effects by orders of magnitude cannot be checked by comparison of theory with experiment.‡

Other types of moving sources

If a moving source has a given velocity potential, its pressure field is proportional to a time derivative of the potential field in a co-ordinate system fixed in the ambient fluid. The *pressure* field of the conventional moving source corresponds to the potential field of the modified source. The moving source used by Lighthill (1962) apparently is a third type. Its pressure field corresponds to the potential field of the conventional source, and it could presumably be created from standing waves of fluid particle acceleration. A source moving with the local fluid in a jet does *not* radiate like the source used by Lighthill in the low frequency limit (as sometimes assumed), but radiates as the modified moving source.

† More precisely a balloon of zero mean diameter.

‡ A referee informs us that these statements have other support. However such support is irrelevant to the subject of this paper and will not be discussed here.

Appendix

Preliminary development

The linearized wave equation governing the propagation of sound waves in a homogeneous fluid is

$$\nabla^2\phi = (1/c^2)\,\phi_{tt}, \tag{1}$$

where ∇^2 is the Laplacian operator, ϕ is a velocity potential, c is the speed of sound and t is time. In rectangular co-ordinates x, y, z

$$\phi_{xx}+\phi_{yy}+\phi_{zz}-(1/c^2)\,\phi_{tt} = 0 \tag{2}$$

and the velocity components u, v, w in the x, y, z directions are equal to ϕ_x, ϕ_y, ϕ_z, respectively.

To permit the construction of singularities in the x, y plane the fluid is divided into an upper and a lower region separated by the x, y plane. In each region an equation similar to (2) must be satisfied. On the x, y plane the pressure in the upper fluid must match the pressure in the lower fluid at every point. The continuity equation is not satisfied across the x, y plane in general, and this failure to satisfy continuity corresponds to the presence of sources in the x, y plane. Let $\eta_u(x,y)$ be the vertical displacement of the lower boundary of the upper fluid and $\eta_l(x,y)$ be the vertical displacement of the upper boundary of the lower fluid. Then $(\eta_u-\eta_l) = \Delta\eta$ indicates the presence of sources. More precisely the local stationary source strength per unit area on the x, y plane is $[\partial\Delta\eta/\partial t]_{x\,\text{const}}$ (i.e. $\Delta\phi_z$ at $z = 0$).

Singularities such as a line source or point source are constructed by superposition of periodic source distributions in the x, y plane (i.e. the use of Fourier methods). For stationary sources the periodic source distributions needed correspond to standing waves of $\partial\Delta\eta/\partial t$ in the x co-ordinate system. Where line or point sources are moving with velocity V_s a new co-ordinate, $\xi = x-V_st$, can be used.

It is worth noting that in the ξ co-ordinate system standing waves of fluid surface acceleration, fluid surface velocity and fluid surface displacement amount to the same thing and are equivalent to standing waves of fluid *particle* displacement. Standing waves of fluid *particle* velocity and fluid *particle* acceleration are entirely different from the above and different from each other. It is merely necessary to distinguish carefully between fluid particles and the surface containing them.

One might expect the periodic source distributions required to correspond to waves of $(\Delta\eta_t)_{\xi\,\text{const}}$ (waves of the fluid surface) which are 'standing' waves in the uniformly moving co-ordinate system. Such a source has the conceptual advantage that the fluid introduced travels with the source and is kept separate from the surrounding fluid. This is consistent (within small perturbation restrictions) with the picture of an expanding and contracting balloon in uniform motion. However, this does not correspond to the conventional moving source, but to a modified moving source.

The conventional moving source is constructed from waves of $(\Delta\eta_t)_{x\,\text{const}}$ (i.e. waves of particle velocity at $z = 0$) which are standing waves in the uniformly moving co-ordinates. This source introduces fluid which is then left behind, and later removes different fluid particles from the surrounding medium. This is not consistent with the simple balloon picture but is mathematically simpler and is generally used.

Solutions of (2) are found by separation of variables and correspond to plane waves (or combinations of them) except where there is exponential variation with z. Only those plane wave solutions corresponding to outwardly moving waves (waves moving up in the upper region and down in the lower region) are needed to construct a source. Inwardly moving waves produce an energy *sink* which is not desired. Among the exponential solutions only those decreasing outwardly need be retained.

Construction of conventional and modified moving sources

A point source is to be constructed on the x axis moving with velocity V_s less than c in the positive x direction. Since there is symmetry about the x, y plane the pressure condition is satisfied automatically and only the upper fluid need be considered. For this region, separation of variables and choice of up-moving waves gives a solution of the wave equation as

$$\begin{aligned}\phi = {} & B\cos(k_2 y)\cos[k_1(x-V_s t)-\omega t+z\{(\omega'+k_1M)^2-k_1^2-k_2^2\}^{\frac{1}{2}}] \\ & +C\cos(k_2 y)\cos[k_1(x-V_s t)+\omega t-z\{(\omega'-k_1M)^2-k_1^2-k_2^2\}^{\frac{1}{2}}], \qquad (3)\end{aligned}$$

where $(\omega'\pm k_1M)^2 > (k_1^2+k_2^2)$ and $\omega' = \omega/c$, $M = V_s/c$. k_1, k_2 are wave-numbers and ω is the frequency.

$\xi = x - V_s t$ is a distance measured in the x direction but in the moving co-ordinate system. Referring to the moving co-ordinates it appears that $(k_1\xi-\omega t)$ indicates a wave moving in the positive ξ direction, while $(k_1\xi+\omega t)$ corresponds to a wave moving in the opposite direction. It is the superposition of such waves that produces a standing wave in the ξ co-ordinate.

The rate of displacement of the lower boundary of the upper fluid region is η_t. By symmetry $\Delta\eta_t = 2\eta_t$. For the conventional moving source $\eta_t = (\partial\eta/\partial t)_{x\,\mathrm{const}}$. For the modified moving source $\eta_t = (\partial\eta/\partial t)_{\xi\,\mathrm{const}}$ and the terms in braces $\{\ \}^*$ in the following equations must be included. These terms are omitted for the conventional source.

$$\begin{aligned}\Delta\eta_t = {} & -2B\{(\omega'+k_1M)^2-k_1^2-k_2^2\}^{\frac{1}{2}}\cos(k_2y)\sin(k_1\xi-\omega t)\left\{\frac{\omega'}{\omega'+k_1M}\right\}^* \\ & +2C\{(\omega'-k_1M)^2-k_1^2-k_2^2\}^{\frac{1}{2}}\cos(k_2y)\sin(k_1\xi+\omega t)\left\{\frac{\omega'}{\omega'-k_1M}\right\}^*. \qquad (4)\end{aligned}$$

In order that $\Delta\eta_t$ be a standing wave in the ξ co-ordinate the two waves moving in opposite directions must have the same amplitude; hence

$$\frac{C}{B} = \frac{\{(\omega'+k_1M)^2-k_1^2-k_2^2\}^{\frac{1}{2}}}{\{(\omega'-k_1M)^2-k_1^2-k_2^2\}^{\frac{1}{2}}}\left\{\frac{(\omega'-k_1M)}{(\omega'+k_1M)}\right\}^*. \qquad (5)$$

If for convenience
$$B = \frac{A(k_1,k_2)\{\omega'+k_1M\}^*}{\{(\omega'+k_1M)^2-k_1^2-k_2^2\}^{\frac{1}{2}}}, \qquad (6)$$

then
$$C = \frac{A(k_1, k_2)\{\omega' - k_1 M\}^*}{\{(\omega' - k_1 M)^2 - k_1^2 - k_2^2\}^{\frac{1}{2}}} \tag{7}$$

and for $(\omega' \pm k_1 M)^2 > (k_1^2 + k_2^2)$ (condition satisfied for both signs)

$$\phi = \frac{A(k_1, k_2) \cos [k_1 \xi - \omega t + z\{(\omega' + k_1 M)^2 - k_1^2 - k_2^2\}^{\frac{1}{2}}] \cos (k_2 y) \{\omega' + k_1 M\}^*}{\{(\omega' + k_1 M)^2 - k_1^2 - k_2^2\}^{\frac{1}{2}}}$$
$$+ \frac{A(k_1, k_2) \cos [k_1 \xi + \omega t - z\{(\omega' - k_1 M)^2 - k_1^2 - k_2^2\}^{\frac{1}{2}}] \cos (k_2 y) \{\omega' - k_1 M\}^*}{\{(\omega' - k_1 M)^2 - k_1^2 - k_2^2\}^{\frac{1}{2}}}. \tag{8}$$

Similarly it can be shown that when $(k_1^2 + k_2^2) > (\omega' \pm k_1 M)^2$

$$\phi = \frac{A(k_1, k_2) \sin (k_1 \xi - \omega t) \exp [-z\{k_1^2 + k_2^2 - (\omega' + k_1 M)^2\}^{\frac{1}{2}}] \cos (k_2 y) \{\omega' + k_1 M\}^*}{\{k_1^2 + k_2^2 - (\omega' + k_1 M)^2\}^{\frac{1}{2}}}$$
$$- \frac{A(k_1, k_2) \sin (k_1 \xi + \omega t) \exp [-z\{k_1^2 + k_2^2 - (\omega' - k_1 M)^2\}^{\frac{1}{2}}] \cos (k_2 y) \{\omega' - k_1 M\}^*}{\{k_1^2 + k_2^2 - (\omega' - k_1 M)^2\}^{\frac{1}{2}}}. \tag{9}$$

When neither condition is satisfied for both signs, ϕ is composed of one term from (8) and one term from (9) chosen in the appropriate manner.

It is now necessary to determine $A(k_1, k_2)$ so that $\Delta\eta_t$ has the form of a δ-function at $\xi = 0$, $y = 0$ (i.e. $\Delta\eta_t$ is proportional to the product of a δ-function in ξ times a δ-function in y).

Replacing B and C in (4) from (6) and (7) gives for $(\omega' \pm k_1 M)^2 > (k_1^2 + k_2^2)$

$$\Delta\eta_t = 4A(k_1, k_2) \cos (k_2 y) \cos (k_1 \xi) \sin (\omega t) \{\omega'\}^*. \tag{10}$$

By differentiating (8) and (9) with respect to z and setting $z = 0$, it can be checked that the above expression for $\Delta\eta_t$ actually applies for all values of $(k_1^2 + k_2^2)$. Integrating over all k_1 and k_2 gives

$$\Delta\eta_t = 4\{\omega'\}^* \sin (\omega t) \int_0^\infty \int_0^\infty A(k_1, k_2) \cos (k_2 y) \cos (k_1 \xi)\, dk_1\, dk_2. \tag{11}$$

$\Delta\eta_t$ is the source strength and in order that it should have the form of a δ-function we require (Sommerfeld 1949, p. 298)

$$A(k_1, k_2) = \nu / 4\pi^2 \{\omega'\}^*, \tag{12}$$

where ν is the source strength (i.e. the maximum volume of fluid introduced per unit time).

$A(k_1, k_2)$ is now determined by (12) and ϕ can be determined by inserting the expression for $A(k_1, k_2)$ in (8) and (9) and integrating over the first quadrant of the k_1, k_2 plane (zero to infinity in k_1 and k_2). The proper expressions for ϕ must of course be chosen for the different regions in this quadrant, defined by

$$(k_1^2 + k_2^2) = (\omega' + k_1 M)^2 \quad \text{and} \quad (k_1^2 + k_2^2) = (\omega' - k_1 M)^2.$$

In order to simplify the presentation of the integrals it is convenient to extend the integration in k_1 and k_2 over the entire k_1, k_2 plane ($-\infty$ to $+\infty$ in k_1 and k_2).

Then, using the complex form,

$$\phi = \frac{\nu}{8\pi^2}\mathscr{R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp i[k_1\xi + k_2 y - \omega t + z\{(\omega' + k_1 M)^2 - k_1^2 - k_2^2\}^{\frac{1}{2}}]}{\{(\omega' + k_1 M)^2 - k_1^2 - k_2^2\}^{\frac{1}{2}}} \times \left\{\frac{\omega' + k_1 M}{\omega'}\right\}^* dk_2\, dk_1, \quad (13)$$

where $\mathscr{R}$ denotes 'real part of'.

This is the velocity potential for a point source of strength ν on the x axis moving at velocity $V_s = Mc$ in the positive x direction. The term in braces, $\{(\omega' + k_1 M)/\omega'\}^*$, appears only for the modified source.

In cylindrical co-ordinates the moving point source can be constructed by superimposing cylindrical standing waves along a cylinder parallel to the x (or ξ) direction. If the source strength is chosen as a δ-function in ξ a ring source is formed. The radius of the ring is then allowed to approach zero, keeping constant the total source volume emitted, and the expression for the moving point source in cylindrical co-ordinates is obtained as

$$\phi = \frac{\nu}{8\pi}\mathscr{R}\int_{-\infty}^{\infty}\exp i[k(x - V_s t) - \omega t]\left\{\frac{\omega' + kM}{\omega'}\right\}^* H_0^{(1)}(r\{(\omega' + kM)^2 - k^2\}^{\frac{1}{2}})\, dk. \quad (14)$$

Again, the term in braces appears only for the modified source. Here $r = (y^2 + z^2)^{\frac{1}{2}}$, k is the wave-number, and $H_0^{(1)}$ denotes the Hankel function of the first kind.

Evaluation of integrals in the far field

The integrals can be evaluated in the far field by standard methods such as the saddle-point method or the method of stationary phase. This is tedious but straight-forward, and we merely give the results.

$$\phi = -\frac{\nu}{4\pi R}\sin\left[\omega t - \frac{M\omega'\xi}{\beta'^2} - \frac{\omega' R}{\beta'^2}\right]\left\{\frac{1 + (M\xi/R)}{\beta'^2}\right\}^*, \quad (15)$$

where

M = Mach number ($M < 1$),

$\beta'^2 = 1 - M^2$,

$R = \{\beta'^2(y^2 + z^2) + \xi^2\}^{\frac{1}{2}}$.

Without the term $\{[1 + (M\xi/R)]/\beta'^2\}^*$ this corresponds to the expression given by Garrick (1957) for a subsonic moving source of conventional type. If $M \to 0$ the two types of moving sources merge to become the simple stationary source since R then becomes a true radius.

It is interesting to place the source in a jet so that it is stationary relative to the ambient air outside the jet. (Such a situation might be checked experimentally (see Atvars *et al.* 1965) if the experimental source-type were known and comparable.) To accomplish this theoretically we go back to the integral form of the source, (14) in cylindrical co-ordinates, and introduce reflected waves within the jet so that for each wave-number the perturbation pressures and displacements on either side of the jet boundary can be equated at the boundary. This is a standard procedure, and subsequent evaluation of far field pressures by saddle-point or stationary phase methods also is standard. However both of these operations are tedious and lengthy. In this note we have presented only the low frequency limit results shown in figure 1.

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